A Stability Criterion for Stochastic Hybrid Systems

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Abstract

This paper investigates the notion of stability for Stochastic Hybrid Systems. The uncertainty is introduced in the discrete jumps between the domains, as if we had an underlying Markov Chain. The jumps happen every fixed time T; moreover, a result is given for the case of probabilistic dwelling times inside each domain. Unlike the more classical Hybrid Systems setting, the guards here are time-related, rather than space-related. We shall focus on vector fields describing input-less dynamical systems. Clearly, the uncertainty intrinsic to the model forces to introduce a fairly new definition of stability, which can be related to the classical Lyapunov one though. Proofs and simulations for our results are provided, as well as a motivational example from finance.

1 Introduction

1.1 Motivations

Hybrid systems have been extensively studied in the past decade, both concerning their theoretical framework, as well as relating to the increasing number of applications they are employed for [2,3]. However, the subfield of Stochastic Hybrid Systems (*SHS*) is fairly young. There are intuitively two ways to introduce uncertainty in the traditional Hybrid Systems' framework. The first one does so in the continuous-time dynamics through the use of Stochastic Differential equations, rather than the classical *ODE*'s [4]. The second way is to embed the randomness into the discrete jumps [5]: this can be done, for instance, through the use of a transition probability matrix. If these probabilities are independent of the points in the domains where the jumps may occur, then we might think of an underlying Markov Chain. In this last case, which will also be the one we shall be focusing on in the sequel, we can think of having a "Hybrid

Markov Chain". Having adopted this last framework, the paper investigates some stability issues. Even if the concept of stability has already been tackled in a deterministic setting, the literature appears to be rather sparse for the Stochastic case. Some results have been achieved for the simplified setting of Markov-Jumps-Linear-Systems, which can be thought of as a simplified setting. Under this framework, we shall extend and generalize these results. The motivations for this work are manifold, but mainly the acknowledgment of the limit of the classical deterministic approach for hybrid systems and the need to introduce some uncertainty. Application-wise, along with suggesting a motivational example that deals with Stocks Pricing evolution, we believe these models could have wide application in other fields, the biological one being a notable instance.

The outline of the paper is as follows: after a formal description of the setting, we first introduce a definition of *stability in probability*, motivate and justify it. Then, we propose some theorems for stability, and prove them: to begin with, we shall consider fixed-time jumps, say every T time; then, we will describe the jumping times via probability distributions. An example shows the viability of these criteria. Then, an applicative example is presented. Extensions, remarks and future work conclude the paper.

1.2 Setting

We are given a hybrid system [11], *i.e.* a collection $H = (Q, X, f, Init, D, E, G, R)$, and the specifications are as follows:

- Q: $\{q_1, q_2, ..., q_n\}$ is a finite set of discrete states;
- X: Continuous State with a continuous variable $x \in \mathbb{R}^m$;
- $f : Q \times X \to \mathbb{R}^m; \dot{x} = f(q_i, x)$ is the vector field related to node q_i[1];
- $Init = Q \times X$ is the set of initial states;
- $D : Q \to P(X)$: a compact subset in $\mathbb{R}^m$, comprehensive of the origin (the "domain")[2];
- E: a set of edges, which in this case are not "spacial";
- $G : E \to P(X)$: the "guard"; here we shall consider 2 cases:
  1. After time T the continuous state jumps;
  2. The jumps are random in time, in the sense that the dwelling times are described by *i.i.d.* random variables, which we assume to have finite mean;
- $R : E \to P(X)$: The reset map is a general function with bounded Lipschitz constant for every node. In our setting, the discrete jumps occur according to a Markov transition matrix $[P_{ij}]$; moreover, the embedded Markov Chain is supposed to be *irreducible*[3] and *positive recurrent*[4] (a sufficient condition would be $P_{ij} \neq 0 \ \forall i, j$).

[1] Often, we will use the simpler notation $f^i(x)$, rather than $f(q_i, x)$

[2] Here $P(X)$ is the power set (the set of all the subsets) of X.

[3] All the pairs of states communicate.

[4] The return time to each node is finite. Actually, given that the chain is a recurrent class, the positiveness is due to its finiteness.

We furthermore underline the following remarks:

- We focus on the evolution of the system for a long time, such that the Markov Chain can get to the steady state: this idea will drive all the proofs;
- The transition probability matrix is independent of the point where the jump occurrs, *i.e.* we have just one of such matrices for every domain;
- The Hybrid MC is assumed to be *non blocking*, and we exclude the presence of the *Zeno behavior*;
- As already described, the domains have no geometrical guards, in other words they can be considered unbounded;
- All the domains share the same equilibrium point, say the origin, and can be linearized around it.

As an example, please refer to Figure 1.

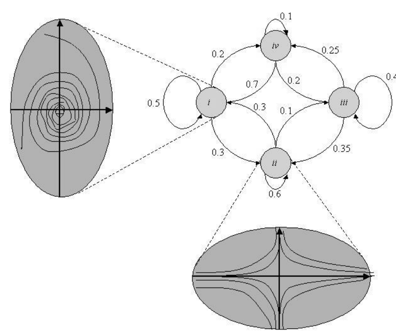

Figure 1: A simple pictorial example for a Stochastic Hybrid Sysyem.

2 Criteria for SHS Stability

2.1 Definition of Stability

The mathematical underpinnings of the criteria we will introduce in this paper come both from the classical theory of (deterministic) Hybrid Systems and from Probability Theory. Clearly, in this new setting, the intrinsic notion of *stability* has to be revised, as we have to deal with purely *non-deterministic* systems.

Literature already presents the notion of *stochastic stability* [6], usually referred to a stochastic process described by a stochastic differential equation: here stability is intended in an asymptotical sense, as it is meant to be enforced probabilistically. This naturally leads to equate the concept of stability to that of convergence, at the limit, to an equilibrium point, with probability one.

Having said this, it should be clear how we can give equivalent definitions for stability of an equilibrium point either resorting to a Lyapunov-like argument, or utilizing a more probabilistic flair.

Assume we have a finite number n of domains, with the origin as a common unique equilibrium and with general nonlinear vector fields $f^i, \quad i = 1, ..., n$. Let us furthermore assume that the underlying Markov Chain is irreducible, positive recurrent and with a stable distribution[5].

Definition 2.1. *Given an equilibrium point q in a vector field, we say that the point is (asymptotically)* ***stable in probability*** *if, calling $X(t)$ a realized trajectory of the field,*

$$\lim_{t \to \infty} P(|X(t) - q| > \epsilon) = 0, \quad \forall \epsilon > 0^6.$$

This definition is actually equivalent to the following:

Definition 2.2. *Given an equilibrium point q in a vector field, we say that the point is (asymptotically)* ***stable in probability*** *if, for every region $\mathcal{D}$ containing q, there exist a time τ and a neighborhood $\mathcal{U}$ ($\mathcal{U} \subset \mathcal{D}$) such that every (random) trajectory starting inside $\mathcal{U}$ will be contained in $\mathcal{D}$, for $t > \tau$, with probability 1.*

Remarks: The only distinguishable difference with the classical notion of stability in the Lyapunov sense is the presence of time τ, which is intuitively referred to the concept of steady-state, and is due to the probabilistic flair of this setting. Unlike the deterministic case, here we have the possibility that an expanding domain is visited repeatedly; thus, it may steer *momentarily* the dynamics far away from the equilibrium; in the long run, though, the overall behavior of the hybrid system may bring a contraction. This "overall behavior" is more properly described by the *steady* state of the embedded Markov Chain. The introduction of the first definition is justified by its easiness to be handled in the coming demonstrations, while the second definition helps in relating the concept to a classical control argument.

2.2 Fixed Dwelling Times

In this section, we shall introduce the first criterion for stability. The peculiar assumption here is about the temporal guards of the SHS:

- $G : E \to P(X)$: After time T the continuous state jumps.

Theorem 2.3. *Let q be an isolated equilibrium of a* hybrid Markov Chain *with n domains, corresponding vector fields f^i, associated flows φ^i and reset maps R_{ij}, $i = 1, \ldots, n; j \leq n$, and let the hybrid MC verify the hypotheses listed in the beginning of the section and π be the steady state distribution of the embedded chain. Let us define:*

- $\nu = \prod_{i=1}^{n} \mathrm{Lip}(\varphi_T^i)^{\pi_i}$;

- $\mu = \prod_{i,j=1}^{n} \mathrm{Lip}(R_{ij})^{\pi_i P_{ij}}$,

that is products within the Lipschitz constants of the flows of the vector fields and of the reset maps associated to each domain.

If the product $\nu\mu < 1$, then the equilibrium q is stable in probability.

[5]There exists one unique steady-state distribution.

[6]The probability space here is the one related to a trajectory $X(t)$, which is a random variable in our setting.

Remarks: The Lipschitz constants of respectively the flows φ^i of the vector fields f^i and of the reset maps R_{ij} account for the contractive or expansive qualities that these functions generate. The steady state probability π, i.e. the vector $\pi = \pi P$, tallies how long the dynamics will evolve in each domain; moreover, given the steady state probability in a certain domain i (π_i) and the probability to jump in another domain j (P_{ij}), the likelihood to pass through the reset map R_{ij} has to be averaged out through the factor $\pi_i P_{ij}$.

Proof: Without loss of generality, we shall say that $q = 0$[7].Let's assume we start at $x(0)$ and let $|x(0)| \neq 0$ (the case $|x(0)| = 0$ is trivial, being the origin an equilibrium point). According to the definition of Lipschitz constant, we have

$$|\varphi_t^i(x(0)) - \varphi_t^i(0)| \leq \mathrm{Lip}(\varphi_t^i)|x(0) - 0|$$

so

$$\frac{|\varphi_t^i(x(0))|}{|x(0)|} \leq \mathrm{Lip}(\varphi_t^i)$$

Then, after N jumps, considering only the final point in the hybrid flow, which is a random quantity, we have

$$\left|\frac{x(NT)}{x(0)}\right| \leq \prod_{i=1}^{n}(\mathrm{Lip}(\varphi_T^i))^{n_i} \prod_{j=1}^{n}(\mathrm{Lip}(R_{ij}))^{m_j} \tag{2.1}$$

where $\sum_{i=1}^n n_i = N$, $\sum_{j=1}^n m_j = N$. The problem here is that the quantities n_i and m_j are unknown, being the MC intrinsically non deterministic. The only thing we can state is a relation between n_i and m_i at the limit, *i.e.* when the chain is in steady state; more precisely,

$$\lim_{N\to\infty} \mathrm{Prob}\{n_i = N\pi_i\} = 1$$

Moreover,

$$\lim_{N\to\infty} \mathrm{Prob}\{m_j = \sum_{i=1}^{n} P_{ij} n_i\} = 1$$

hence we get

$$\lim_{N\to\infty} \mathrm{Prob}\{m_j = N\sum_{i=1}^{n} P_{ij} \pi_i\} = 1$$

Therefore, given those last relations and (2.1), it is easy to set up an inequality for probabilities and consider the following limit:

$$\begin{aligned} \lim_{N\to\infty} \mathrm{Prob}\Big\{ |x(NT)| > \epsilon' \Big\} &= \lim_{N\to\infty} \mathrm{Prob}\Big\{ \left|\frac{x(NT)}{x(0)}\right| > \epsilon \Big\} \\ &\leq \lim_{N\to\infty} \mathrm{Prob}\Big\{ \prod_{i=1}^{n}(\mathrm{Lip}(\varphi_T^i))^{n_i} \prod_{j=1}^{n}(\mathrm{Lip}(R_{ij}))^{m_j} > \epsilon \Big\}, \end{aligned}$$

where we considered $\epsilon = \frac{\epsilon'}{|x(0)|}$. Substituting through the relations before, we obtain,

[7]In the more general case of equilibrium q different from the origin, it is possible to operate a smooth transformation of coordinates; this does not affect the thread of the argument.

$$\lim_{N\to\infty} \text{Prob}\Big\{ \prod_{i=1}^{n} (\text{Lip}(\varphi_T^i))^{N\pi_i} \prod_{j=1}^{n} (\text{Lip}(R_{ij}))^{N\sum_{i=1}^{n} P_{ij}\pi_i} > \epsilon \Big\} =$$

$$= \lim_{N\to\infty} \text{Prob}\Big\{ \prod_{i=1}^{n} (\text{Lip}(\varphi_T^i))^{N\pi_i} \prod_{i,j=1}^{n} (\text{Lip}(R_{ij}))^{NP_{ij}\pi_i} > \epsilon \Big\}$$

We can bring the limit inside the expression and, after some arrangement, we get:

$$\text{Prob}\Big\{ \lim_{N\to\infty} [\prod_{i=1}^{n} (\text{Lip}(\varphi_T^i))^{\pi_i} \prod_{i,j=1}^{n} (\text{Lip}(R_{ij}))^{P_{ij}\pi_i}]^N > \epsilon \Big\} = \text{Prob}\Big\{ \lim_{N\to\infty} [\nu\mu]^N > \epsilon \Big\}$$

Therefore, if $\nu\mu < 1$, we have

$$\lim_{N\to\infty} (\nu\mu)^N = 0$$

hence,

$$\lim_{N\to\infty} \text{Prob}\{ \left| \frac{x(NT)}{x(0)} \right| > \epsilon \} = 0, \ \forall \epsilon > 0$$

i.e. , we have the asymptotical convergence with probability one:

$$\lim_{N\to\infty} \text{Prob}\{ |x(NT) - x(0)| > \epsilon' \} = 0, \ \forall \epsilon' > 0$$

QED

Remarks: The condition in this theorem is clearly just a sufficient one. It turns out that for linear and scalar systems, it is necessary as well. In general, there may be pathological cases where $\nu\mu \geq 1$, but the equilibrium q is still *stable in probability*; nevertheless, it looks like the hypotheses are rather *tight*, at least in the linear case.

2.3 Probabilistic Dwelling Times

After introducing the previous sufficient condition, under the hypothesis of fixed-time jumps, let us make a different assumption on the guards:

- $G : E \to P(X)$: The jumps are random, in the sense that the dwelling times are described by *i.i.d.* random variables.

In other words, we are introducing a new type of uncertainty on the time the trajectory is spending inside a domain, along with the randomness related to the discrete jumps between the domains.

We shall propose an extension to the previous theorem for linear, time-invariant (*LTI*) systems.

Theorem 2.4. *Let q be an isolated equilibrium of a* hybrid Markov Chain *with n domains, corresponding LTI vector fields A^i, associated flows φ^i and reset maps R_{ij}, $i = 1, \ldots, n; j \leq n$. Assume each A^i commutes with its transpose. Let π be the steady state distribution of the embedded chain and the hybrid MC verify the hypotheses listed in the beginning of the section, except for the following:*

- G_i: *the guard of domain* i *is a random arrival-time variable* t_i *with finite expectation* λ_i ($E[t_i] = \lambda_i$).

Let us define:

- $\nu = \prod_{i=1}^{n} \text{Lip}(\varphi^i_{\lambda_i})^{\pi_i}$;

- $\mu = \prod_{i,j=1}^{n} \text{Lip}(R_{ij})^{\pi_i P_{ij}}$,

that is products within the Lipschitz constants of the flows of the vector fields and of the reset maps associated to each domain.

If the product $\nu\mu < 1$, *then the equilibrium* q *is* stable in probability.

Proof: As before, we shall just prove the statement with the origin as the equilibrium point, without any loss of generality; moreover, we will again consider a sequence of N jumps for the hybrid trajectory. Unlike before, though, the final time will be unknown, a random variable itself. The following two relations on the quantities n_i and m_j, defined in the body of the previous proof, are still valid:

$$\lim_{N\to\infty} \text{Prob}\{n_i = N\pi_i\} = 1$$

$$\lim_{N\to\infty} \text{Prob}\{m_j = N\sum_{i=1}^{n} P_{ij}\pi_i\} = 1$$

So, we have knowledge, at the limit, of the fraction of jumps spent into a node, but not the cumulative amount of time the trajectory stayed in that node. In node i, given that the flow passed n_i times through it, we have a cumulative time of $\sum_{l=1}^{n_i} t_l$, where t_l are *i.i.d.* random variables. Therefore, the total amount of time for the hybrid flow is $\sum_{i=1}^{n}\sum_{l_i=1}^{n_i} t_{l_i}$, where, as usual, $\sum_{i=1}^{n} n_i = N$.

Proceeding as before, we get to have:

$$\left|\frac{x(\sum_{i=1}^{n}\sum_{l_i=1}^{n_i} t_{l_i})}{x(0)}\right| \quad \leq \quad \prod_{i=1}^{n}\prod_{l_i=1}^{n_i} \text{Lip}\varphi^i_{t_{l_i}} \prod_{j=1}^{n} (\text{Lip}(R_{ij}))^{m_j} \qquad (2.2)$$

The contribute of the reset functions is the same as before, as they contain no continuous-time dynamics within themselves. Due to the properties of the state transition matrix of LTI systems[8] , we observe that:

$$\text{Lip}(\varphi^i_{t_j-t_k})\text{Lip}(\varphi^i_{t_k}) = \text{Lip}(\varphi^i_{t_j}), \forall i, \forall t_j > t_k.$$

As a consequence, we have that

$$\text{Lip}(\varphi^i_{Mt_k}) = (\text{Lip}(\varphi^i_{t_k}))^M, \forall i, \forall t_k, M \in \mathbb{N}^+.$$

Then, we can write

[8] Here we exploit the following facts: we conceive the induced norm of a matrix as the spectral one, $||e^{At_1}|| = max\{|\lambda| : \lambda \in \sigma((e^{At_1})^T(e^{At_1}))\} = max\{|\lambda| : \lambda = e^{\mu t_1}, \mu \in \sigma(A^T + A)\}$ if A commutes with A^T, and where $\sigma(B) = Spectrum(B)$, for any matrix B. Then, through direct manipulation, we get that in this special case $||e^{At_1}||||e^{At_2}|| = ||e^{A(t_1+t_2)}||$.

$$\left|\frac{x(\sum_{i=1}^{n}\sum_{l_i=1}^{n_i} t_{l_i})}{x(0)}\right| \leq \prod_{i=1}^{n} \text{Lip}\varphi^i_{\sum_{l_i=1}^{n_i} t_{l_i}} \prod_{j=1}^{n} (\text{Lip}(R_{ij}))^{m_j}$$

Now, let's notice that within each node, we have the sum of independent random variables t_i, identically distributed within a node; If we call $s_n = \sum_{i=1}^{n} t_i$, then, by the *weak law of large numbers*,

$$\lim_{n\to\infty} P\Big(\Big|\frac{s_n}{n} - E[t]\Big| > \epsilon\Big) = 0, \quad \forall \epsilon \geq 0.$$

In other words, the variable $\frac{s_n}{n}$ tends to the expectation of the variables, which in node i we denoted as λ_i.

Consequently, passing at the limit, we conclude that

$$\begin{aligned}\lim_{N\to\infty} P\Big(\left|\frac{x(\sum_{i=1}^{n}\sum_{l_i=1}^{n_i} t_{l_i})}{x(0)}\right| > \epsilon\Big) &\leq \lim_{N\to\infty} P\Big(\prod_{i=1}^{n}(\text{Lip}(\varphi^i_{\lambda_i}))^{n_i} \prod_{j=1}^{n}(\text{Lip}(R_{ij}))^{m_j} > \epsilon\Big)\\ &= \lim_{N\to\infty} P\Big(\Big[\prod_{i=1}^{n}(\text{Lip}(\varphi^i_{\lambda_i}))^{\pi_i} \prod_{i,j=1}^{n}(\text{Lip}(R_{ij}))^{P_{ij}\pi_i}\Big]^N > \epsilon\Big)\end{aligned}$$

Therefore, if the quantity

$$\nu\mu = \prod_{i=1}^{n}(\text{Lip}(\varphi^i_{\lambda_i}))^{\pi_i} \prod_{i,j=1}^{n}(\text{Lip}(R_{ij}))^{P_{ij}\pi_i} < 1,$$

then

$$\lim_{N\to\infty} P\Big(\left|\frac{x(\sum_{i=1}^{n}\sum_{l_i=1}^{n_i} t_{l_i})}{x(0)}\right| > \epsilon\Big) = 0.$$

QED

Special Case As a special case, we can consider the t_i to be exponential random variables with rate λ_i. Then, the *renewal process* associated with the jumps between the nodes is actually a *Poisson process*.

2.3.1 A more compact generalization

Thus far, we have introduced uncertainty within our setting at two levels: first, in the transition probabilities between nodes; second, in the dwelling times in each node. It is possible to blend these two factors into a unique item considering a continuous Markov Chain setting. Let us refer to a finite space $S = \{1, 2, \ldots, n\}$; a Markov process $B(t), t > 0$ is a continuous time Markov Chain if

$$P_{ij}(t) = P\{B(t+s) = j | B(s) = i\}, \forall s \geq 0.$$

From the Markov hypothesis we have that the following must hold:

- $P_{ij} \geq 0$;

- $\sum_{j=0}^{n} P_{ij}(t) = 1, \quad i,j = 0,1,\ldots,n;$
- $P_{ik}(s+t) = \sum_{j=0}^{n} P_{ij}(s)P_{jk}(t)$, for $s,t \geq 0$;
- $\lim_{t\to 0^+} P_{ij}(t) = \begin{cases} 1, & i=j \\ 0, & i \neq j. \end{cases}$

The third property tells that P is continuous; moreover, it turns out that it is differentiable as well; this enables to calculate the intensities that describe the sojourn times within each domain.

$$\lim_{h\to 0^+} \frac{1-P_{ii}(h)}{h} = q_i,$$
$$\lim_{h\to 0^+} \frac{P_{ij}(h)}{h} = q_{ij}, \quad i \neq j,$$

where $0 \leq q_{ij} < \infty, i \neq j$ and $0 \leq q_i < \infty$. Then, from the second property, we derive that

$$q_i = -\sum_{j=1, j\neq i}^{n} q_{ij}.$$

The rates can be interpreted as the generator of the MC, as in the following:

$$\begin{aligned} P\{B(t+h) = j | B(t) = i\} &= q_{ij}h + o(h), \quad i \neq j, h \downarrow 0; \\ P\{B(t+h) = i | B(t) = i\} &= 1 - q_i h + o(h), h \downarrow 0. \end{aligned}$$

The probability to jump from node i within $(t, t+h]$ is then proportional to $\sum_{j\neq i} q_{ij}$. The dwelling time for node i then has a duration that is exponentially distributed with parameter q_i.

Conversely, it could be possible to start the definition from the intensities, and then derive the jump probabilities; these will be only constrained to assume precise values near time 0.

This way, it is possible to reframe our setting more synthetically within the continuous-time Markov Chain theory; clearly, all the former result can be easily tailored to this framework.

2.4 Simulations and Results

We have tested the sufficient condition in a simplistic case, a stochastic hybrid system with 5 nodes, and fixed-time jumps. We assumed as transition probability matrix :

$$P = \begin{pmatrix} 0.3 & 0.2 & 0.5 & 0 & 0 \\ 0 & 0.4 & 0 & 0 & 0.6 \\ 0 & 0 & 0.4 & 0.6 & 0 \\ 0.5 & 0 & 0.2 & 0 & 0.3 \\ 0 & 0 & 0.7 & 0 & 0.3 \end{pmatrix};$$

the steady state probability vector was computed to be

$$\pi = [0.168, 0.056, 0.392, 0.235, 0.148]$$

We considered simple linear vector fields in $\mathbb{R}^2$. The initial point was taken to be $x_0 = [3, 2]$. For simplicity we have also considered identity reset maps.

The simulations have confirmed the theoretical results; calculating successive powers of the matrix P gave us an empirical assessment of when the steady state occurred. With values of the product $\nu\mu$ very close to 1, we experienced how different realizations could carry different results, but as soon as we increased the number of steps there was a sort of stabilization around a region of points with the same magnitude as the starting one. We plotted three different realizations in these three cases, matching the expected results.

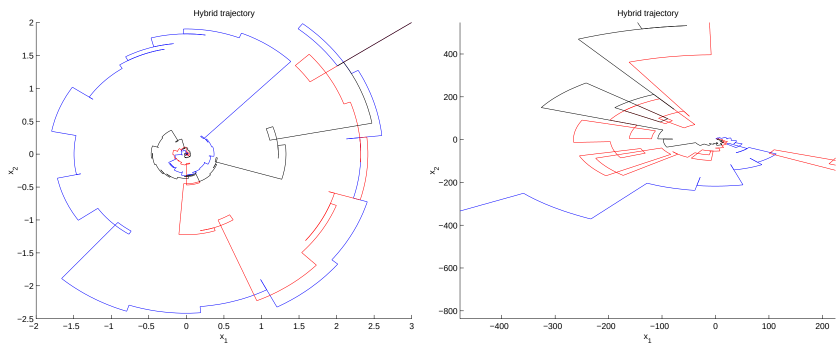

Figure 2: Hybrid trajectories corresponding to $\nu < 1$ and $\nu > 1$

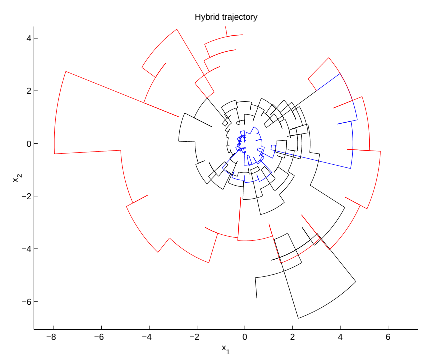

Figure 3: Hybrid trajectories corresponding to $\nu \approx 1$

We have tuned the third node, the most sensible one due to the highest value of the steady state probability, obtaining the following values of ν:

case	ν
1	0.9605
2	1.0635
3	0.9999

Table 2: Values of the decision quantities for system's stability.

3 An Applicative, Motivational Example

3.1 Stocks Pricing

Assume we are analyzing a Competitive Market environment: in this setting, under usual trading conditions, the fair value of the stocks of a company, call it p^*, is automatically attained through the interactions between the stockholders (*i.e.* the sell and purchase of them); on the contrary, a particular situation may overprice their value, or conversely depreciate them.

Let's then imagine we deal with a fixed number of stocks n introduced in a market; moreover, let's say that during a certain time horizon $[t, t + \alpha]$, X stockholders want to make a purchase and Y operators wish to sell their stocks. For simplicity, say that each entity is able to make only one transaction at a time and that $n \gg (X, Y)$, *i.e.* there is plenty of stocks on the market to satisfy any possible demand. Moreover, at each time slot, at most one new entity can be admitted into the market; we can describe the probability of this new entrance so that the inter-arrival time α is then an exponential random variable.

Now, the market may experience three possible different "status quo" (see Figure 4):

- *Equilibrium*: this situation happens when the number of purchasers is equal to that of the operators willing to sell their titles, *i.e.* $X = Y$. In this scenery there is a natural convergence to the fair price p^*, with rate proportional to the actual number of entities currently involved in the deal.
- *Overpricing*: in this framework there are more stock owners willing to buy the equities than holders willing to cease their property, *i.e.* $X > Y$. As intuitive, the price will increase with rate proportional to $X - Y$.
- *Depreciation*: in this case, the number of holders willing to give out their stocks is higher than that of the people agreeing to buy them; in other words, $Y > X$. Therefore, the price will tend to decrease, with rate proportional to $Y - X$.

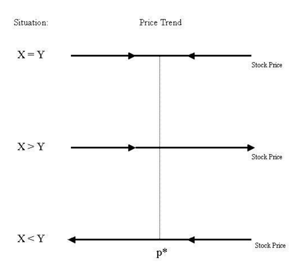

Figure 4: The three possible Market conditions: Equilibrium, Overpricing and Depreciation.

The current market situation can be modelled as described in Figure 4; starting from a certain situation, at each time slot we can increase/decrease/leave

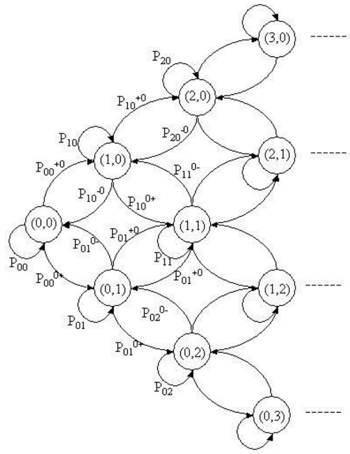

Figure 5: The underlying Birth-Death 2-dimensional Markov Chain. In every node the couple represents the number of respectively buyers(X) and sellers(Y).

unchanged either the number of buyers, or that of vendors; this is described according to some probabilities. In other words, we globally have an underlying *2 dimensional Birth-Death Markov Chain* characterizing the nodes transitions. As already stated, at each jumping time the system switches according to the fixed probability distribution.

When someone wanted to get into the Market, he could use the model to predict the fluctuations of the stocks value, given the current initial condition for the price and the "situation" of the Market itself, that is the likelihood of a modification in the number of buyers and sellers.

Fact 3.1. *The conclusions of Theorems (2.3), (2.4) are not affected in the case of an infinite number n of domains.*

Proof: This *scalability* property follows from the proof of the Theorem, as long as not all the steady-state probabilities of the nodes are equal to zero. Also, it can be intuitively accepted once we understand that all our stability definitions work in the long run, and with asymptotic behavior.

Therefore, given that we shall be able to compute a steady state distribution for the infinite-dimensional birth-death chain, the criterium will still hold true. The computability of the distribution hinges on possible symmetries in the transition probability matrix. In case we shall not be able to compute it, we could still limit the depth of the Markov Chain and apply more orthodox methods, like the cutsets one.

4 Conclusions and Future Work

In this paper, the issue of stability in a special framework for stochastic hybrid systems has been studied. A proper definition of stability in the stochastic sense is given and justified. Stability criteria are proposed in two different settings.

Matlab simulations confirm the validity of the criteria. Moreover, an applicative and viable example contextualizes the theory in the domain of finance.

Future directions of research will be the extension of the validity of the criterion, as well as the assessment of other kinds of possible conditions for stability.

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